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MOTIVIC TORSORS

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ABSTRACT

The torsor $P_\sigma = \mathrm{Hom}^{\otimes}(H_{\mathrm{DR}}, H_\sigma)$ under the motivic Galois group $G_\sigma = \mathrm{Aut}^{\otimes} H_\sigma$ of the Tannakian category $\mathcal{M}_k$ generated by one-motives related by absolute Hodge cycles over a field k with an embedding $\sigma : k \hookrightarrow \mathbb{C}$ is shown to be determined by its projection $P_\sigma \to P_\sigma/G_\sigma^0$ to a $\mathrm{Gal}(\overline{k}/k)$-torsor, and by its localizations $P_\sigma \otimes_k k_\xi$ at a dense subset of orderings ξ of the field k, provided k has virtual cohomological dimension (vcd) one. This result is an application of a recent local-global principle for connected linear algebraic groups over a field k of vcd ≤ 1.

The singular cohomology with coefficients in the field $\mathbb{Q}$ of rational numbers of a smooth projective — even just complete — variety over $\mathbb{C}$ has a ("pure") Hodge structure. Motives with a realization (usually by means of some cohomology theory) which has a pure Hodge structure are called pure motives. Deligne defined in [D-II] a mixed Hodge structure to be a finite dimensional vector space V over $\mathbb{Q}$ with a finite increasing (weight) filtration $W_\bullet$ and a finite decreasing (Hodge) filtration $F^\bullet$ on $V \otimes_{\mathbb{Q}} \mathbb{C}$ such that $F^\bullet$ induces a Hodge structure of weight n on the graded piece $\mathrm{Gr}_n^W V = W_nV/W_{n-1}V$ for each n. Deligne showed in [D-III] that the cohomology $H^*(E(\mathbb{C}), \mathbb{Q})$ of any variety E over $\mathbb{C}$ — not necessarily complete and smooth — carries a natural mixed Hodge structure. Motives with a realization which has a mixed Hodge structure are called mixed motives for emphasize.

Deligne introduced the notion of a one-motive M — as well as its dual $M^\vee$, and Betti: $M(\mathbb{C})_B$, de Rham: $H_{\mathrm{DR}}(M)$, and ℓ-adic: $H_\ell(M)$, realizations — in

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[D-III], §10, as a simple example of a motive whose Betti realization $M(\mathbb{C})_B$ has a mixed Hodge structure, but does not have a Hodge structure. Let σ: $k \hookrightarrow \mathbb{C}$ be an embedding of a field k in the field $\mathbb{C}$ of complex numbers, and $\overline{\sigma}$: $\overline{k} \hookrightarrow \mathbb{C}$ an extension to an algebraic closure $\overline{k}$. Write $\mathrm{Gal}(\overline{k}/k)$ for the Galois group. For a variety E over k, write σE for the $\mathbb{C}$-variety $E \otimes_{k,\sigma} \mathbb{C}$ $(= E \times_{\mathrm{Spec}\, k,\sigma} \mathrm{Spec}\,\mathbb{C})$.

A one-motive over k is a complex $M = [X \xrightarrow{u} E]$ of length one placed in degrees 0 and 1, comprising a semiabelian variety E (namely an extension $1 \to T \to E \to A \to 0$ of an abelian variety A by a torus T) over k, a finitely generated torsion free $\mathrm{Gal}(\overline{k}/k)$-module X, and a $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism $u : X \to E(\overline{k})$. Note that E is a commutative k-group. One-motives include the Artin motives as $[X \to 0]$ and the Tate motive as $[0 \to \mathbb{G}_m]$. We also write $M = (X, A, T, E, u)$, $M \otimes \mathbb{Q}$ for the isogeny class of M, $\sigma M = [X \xrightarrow{u} \sigma E]$ and $\sigma M(\mathbb{C}) = [X \xrightarrow{u} \sigma E(\mathbb{C})]$. A one-motive M has a "weight" filtration:

$$W_0 M = [X \xrightarrow{u} E],\ W_{-1}M = [0 \to E],\ W_{-2}M = [0 \to T],\ W_{-3}M = [0 \to 0],$$

with graded factors

$$\mathrm{Gr}_0 M = X,\ \mathrm{Gr}_{-1} M = (E/T)[-1] = A[-1],\ \text{and}\ \mathrm{Gr}_{-2} M = T[-1].$$

Put $\mathrm{Gr}^W M = [X \xrightarrow{0} A \otimes_k T]$.

The Betti realization $H_\sigma(M) = \sigma M(\mathbb{C})_B$ of a one-motive $M = [X \xrightarrow{u} E]$ over k is the vector space $T_\sigma(M) \otimes \mathbb{Q}$, where the lattice $T_\sigma(M)$ is the fiber product of $\mathrm{Lie}\,\sigma E(\mathbb{C})$ and X over $\sigma E(\mathbb{C})$, namely the pullback of

$$0 \to H_1(\sigma E(\mathbb{C})) \to \mathrm{Lie}\,\sigma E(\mathbb{C}) \xrightarrow{\exp} \sigma E(\mathbb{C}) \to 1 \text{ by } X \xrightarrow{u} \sigma E(\mathbb{C}).$$

It depends on the embedding σ: $k \hookrightarrow \mathbb{C}$. Then $\sigma M(\mathbb{C})_B$ is a mixed Hodge structure $(V, W_\bullet, F^\bullet)$ of type $\{(0,0), (0,-1), (-1,0), (-1,-1)\}$ whose graded parts are $\mathrm{Gr}_0^W V = X \otimes \mathbb{Q}$, polarizable $\mathrm{Gr}_{-1}^W V = H_1(\sigma A(\mathbb{C}), \mathbb{Q})$, and $\mathrm{Gr}_{-2}^W V = H_1(\sigma T(\mathbb{C}), \mathbb{Q}) = X_*(\sigma T) \otimes \mathbb{Q}$; see [D-III], 10.1.3.

Denote by $\mathcal{M}_k$ the Tannakian category (Deligne–Milne [DM], Definition 2.19) generated by the isogeny classes of one-motives over k, in the category $\mathcal{MR}_k$ of mixed realizations (Jannsen [J], 2.1), related by absolute Hodge cycles (Deligne [D2], 2.10, Brylinski [Br], 2.2.5). The objects of $\mathcal{MR}_k$ are tuples

$$H = (H_{\mathrm{DR}}, H_\ell, H_\sigma; I_{\infty,\sigma}, I_{\ell,\overline{\sigma}}),$$

where ℓ ranges over the rational primes, σ over the embeddings $k \hookrightarrow \mathbb{C}$, and $\overline{\sigma}$ over the $\overline{k} \hookrightarrow \mathbb{C}$, described in [J], p. 10. In particular H_{DR} is a finite dimensional

k-vector space with a decreasing (Hodge) filtration $(F^n; n \in \mathbb{Z})$ and an increasing (weight) filtration $(W_m; m \in \mathbb{Z})$; H_ℓ is a finite dimensional $\mathrm{Gal}(\overline{k}/k)$-module over $\mathbb{Q}_l$ with $\mathrm{Gal}(\overline{k}/k)$-equivariant increasing (weight) filtration $W_\bullet$; H_σ is a mixed Hodge structure (over $\mathbb{Q}$), and

$$I_{\infty,\sigma}\colon H_\sigma \otimes \mathbb{C} \tilde{\to} H_{\mathrm{DR}} \otimes_{k,\sigma} \mathbb{C}, \quad I_{\ell,\overline{\sigma}}\colon H_\sigma \otimes \mathbb{Q}_\ell \tilde{\to} H_\ell \quad (\sigma = \overline{\sigma}|k)$$

are the comparison isomorphisms.

The morphisms in $\mathcal{MR}_k$ are tuples $(f_{\mathrm{DR}}, f_\ell, f_\sigma)_{\ell,\sigma}$ described in [J], p. 11. In particular $f_\sigma\colon H_\sigma \to H'_\sigma$ is a morphism of mixed Hodge structures, $f_{\mathrm{DR}}\colon H_{\mathrm{DR}} \to H'_{\mathrm{DR}}$ is k-linear and $f_\ell\colon H_\ell \to H'_\ell$ is a $\mathbb{Q}_\ell$-linear $\mathrm{Gal}(\overline{k}/k)$-morphism, which correspond under the comparison isomorphisms. The category $\mathcal{MR}_k$ is abelian ([J], 2.3), tensor ([J], 2.7) with identity object $\mathbf{1} = (k, \mathbb{Q}_\ell, \mathbb{Q}; \mathrm{id}_{\infty,\sigma}, \mathrm{id}_{\ell,\overline{\sigma}})$, and it has internal $Hom(H, H') \in \mathcal{MR}_k$ for all H, H' in $\mathcal{MR}_k$ (thus

$$\mathrm{Hom}(H'', Hom(H, H')) = \mathrm{Hom}(H'' \otimes H, H')$$

for all H, H', $H'' \in \mathcal{MR}_k$). For example,

$$\begin{aligned} H_{\mathrm{DR}}(Hom(H, H')) &= \mathrm{Hom}_k(H_{\mathrm{DR}}, H'_{\mathrm{DR}}), \\ H_\ell(Hom(H, H')) &= \mathrm{Hom}_{\mathbb{Q}_\ell}(H_\ell, H'_\ell), \\ H_\sigma(Hom(H, H')) &= \mathrm{Hom}_{\mathbb{Q}}(H_\sigma, H'_\sigma). \end{aligned}$$

Hence $\mathcal{MR}_k$ is rigid (each object H has a dual $H^\vee = Hom(H, 1)$).

Defining the space $\mathrm{AHC}(H)$ of absolute Hodge cycles of $H \in \mathcal{MR}_k$ to be the set of $(x_{\mathrm{DR}}, x_\ell, x_\sigma) \in H_{\mathrm{DR}} \times \prod_\ell H_\ell \times \prod_\sigma H_\sigma$ such that $I_{\infty,\sigma}(x_\sigma) = x_{\mathrm{DR}}$, $I_{\ell,\overline{\sigma}}(x_\sigma) = x_\ell$ for all $\sigma, \overline{\sigma}$ with $\overline{\sigma}|k = \sigma$ and $x_{\mathrm{DR}} \in F^0 H_{\mathrm{DR}} \cap W_0 H_{\mathrm{DR}}$ (it is a finite dimensional vector space over $\mathbb{Q}$), one has $\mathrm{Hom}(H, H') = \mathrm{AHC}(Hom(H, H'))$. A Hodge cycle with respect to σ is a tuple $(x_{\mathrm{DR}}, x_\ell) \in H_{\mathrm{DR}} \times \prod_\ell H_\ell$ such that there is $x_\sigma \in H_\sigma$ with

$$I_{\infty,\sigma}(x_\sigma) = x_{\mathrm{DR}}, \quad I_{\ell,\overline{\sigma}}(x_\sigma) = x_\ell, \quad x_{\mathrm{DR}} \in F^0 H_{\mathrm{DR}} \cap W_0 H_{\mathrm{DR}}.$$

Then $\mathcal{MR}_k$ is a Tannakian category neutral over $\mathbb{Q}$, namely a rigid abelian tensor $\mathbb{Q}$-linear category with a $\mathbb{Q}$-valued fiber ([DM], Definition 2.19: exact faithful $\mathbb{Q}$-linear tensor) functors $H_\sigma^\#\colon \mathcal{MR}_k \to \mathrm{Vec}_{\mathbb{Q}}$, $H \mapsto H_\sigma^\#$. The # emphasizes here that the symbol indicates the underlying vector space. In the literature, and in the abstract of this paper, # is omitted to simplify the notations for the reader who knows when H_σ is regarded as a mixed Hodge structure, and when it is regarded only as a vector space.

The mixed realization $H(M)$ of a one-motive M is

$$(H_{\mathrm{DR}}(M), H_\ell(M), H_\sigma(M);\ I_{\infty,\sigma}, I_{\ell,\overline{\sigma}});$$

see [D-III], 10.1.3: the H are H_1. Note that the dual one motive $M^\vee$ (introduced in [D-III], 10.2.11) satisfies $H(M^\vee) = \mathcal{H}om(H(M), \mathbb{Q}(1))$. Hence $H(M)^\vee = H(M^\vee)(-1)$. From now on, by a motive we mean an object in the Tannakian category $\mathcal{M}_k$ generated in $\mathcal{MR}_k$ by one-motives. The functor $H_\sigma^\#$ — which associates to a motive M the vector space underlying the mixed Hodge Betti realization $\sigma M(\mathbb{C})_B$ — is a fiber functor on $\mathcal{M}_k$, making $\mathcal{M}_k$ Tannakian and neutral over $\mathbb{Q}$. Note that an isomorphic — but not canonically — fiber functor is $H_\sigma^\# \operatorname{Gr}^W$. This fiber functor corresponds to a choice of a Levi decomposition of the motivic Galois group; see the end of the 5th paragraph below.

The category $\mathcal{M}_k$ is not semisimple, but it has a semisimple Tannakian full subcategory $\mathcal{M}_k^{\mathrm{red}}$ of motives generated by abelian varieties ($M = [0 \to A]$) and Artin motives ($M = [X \to 0]$) over k, related by absolute Hodge cycles ([DM], Propositions 6.5 and 6.21). Thus it is the subcategory of $\mathcal{MR}_k$ generated by

$$H(A)(= (H_{1,\mathrm{DR}}(A), H_{1,\text{ét}}(A \otimes_k \overline{k}, \mathbb{Q}_\ell), H_1(\sigma A(\mathbb{C}), \mathbb{Q}))$$

of the abelian varieties A over k, and the Artin motives

$$H(X) = X \otimes \mathbf{1} = (X \otimes k, X \otimes \mathbb{Q}_\ell, X \otimes \mathbb{Q}).$$

Note that the realization $H(T)$ of the torus $[0 \to T]$ is the Tate twisted Artin motive

$$X_*(T) \otimes \mathbf{1}(1)(= (X_*(T) \otimes k(1), X_*(T) \otimes \mathbb{Q}_\ell(1), X_*(T) \otimes \mathbb{Q}(1))),$$

where $X_*(T) = \mathcal{H}om(\mathbb{G}_m, T)$ (internal Hom in the category of one-motives). The subcategory $\mathcal{M}_k^{\mathrm{red}}$ of $\mathcal{M}_k$ is also neutral over $\mathbb{Q}$, by the fiber functor $H_\sigma^\#$.

Denote by $\mathcal{M}_k \otimes k$ the category $(\mathcal{M}_k)_{(k)}$ of [DM], Proposition 3.11, obtained on extending coefficients from $\mathbb{Q}$ to k. It is a Tannakian category neutral over k. The functors $H_\sigma^\# \otimes k$ $(: M \mapsto \sigma M(\mathbb{C})_B \otimes k)$ and $H_{\mathrm{DR}}^\#$ on $\mathcal{M}_k \otimes k$ are fiber functors with values in k. The groups $G_\sigma = \operatorname{Aut}^\otimes(H_\sigma^\# \otimes k | \mathcal{M}_k \otimes k)$ and $G_{\mathrm{DR}} = \operatorname{Aut}^\otimes(H_{\mathrm{DR}}^\# | \mathcal{M}_k \otimes k)$ of automorphisms of the fiber functors are affine group schemes over k ([DM], Theorem 2.11 and Proposition 3.11); they are inner forms of each other. Even a conjectural description of these groups is elusive. The functors $H_\sigma^\# \otimes k$ and $H_{\mathrm{DR}}^\#$ define equivalences $\mathcal{M}_k \otimes k \xrightarrow{\sim} \operatorname{Rep}_k G_\sigma$ and $\mathcal{M}_k \otimes k \xrightarrow{\sim} \operatorname{Rep}_k G_{\mathrm{DR}}$ of tensor categories.

Similarly we have the Tannakian category $\mathcal{M}_k^{\mathrm{red}} \otimes k$, which is semisimple and neutral over k by the fiber functors $H_\sigma^{\#} \otimes k$ and $H_{\mathrm{DR}}^{\#}$, the k-groups $G_\sigma^{\mathrm{red}} = \mathrm{Aut}^{\otimes}(H_\sigma^{\#} \otimes k|\mathcal{M}_k^{\mathrm{red}} \otimes k)$ and $G_{\mathrm{DR}}^{\mathrm{red}} = \mathrm{Aut}^{\otimes}(H_{\mathrm{DR}}^{\#}|\mathcal{M}_k^{\mathrm{red}} \otimes k)$, and the equivalences $\mathcal{M}_k^{\mathrm{red}} \otimes k \tilde{\to} \mathrm{Rep}_k G_\sigma^{\mathrm{red}}$ and $\mathcal{M}_k^{\mathrm{red}} \otimes k \tilde{\to} \mathrm{Rep}_k G_{\mathrm{DR}}^{\mathrm{red}}$. Since the category $\mathcal{M}_k^{\mathrm{red}} \otimes k$ is semisimple, [DM], Remark 2.28 implies that G_σ^{red} and $G_{\mathrm{DR}}^{\mathrm{red}}$ are pro-reductive (meaning that the connected component is the projective limit of connected reductive groups). The group G_σ^{red} (resp. $G_{\mathrm{DR}}^{\mathrm{red}}$) is the maximal pro-reductive quotient of the affine group scheme G_σ (resp. G_{DR}).

Note that a $\otimes$-functor $F\colon A \to B$ of Tannakian categories and a fiber functor β on B define a map $f\colon G_B = \mathrm{Aut}^{\otimes}(\beta) \to G_A = \mathrm{Aut}^{\otimes}(\beta \circ F)$ of the motivic groups (the image $g^A = (g^A_{X_A}) = f(g^B)$ is defined by $g^A_{X_A} = g^B_{F(X_A)}$), and vice versa: $f\colon G_B \to G_A$ defines $F\colon A = \mathrm{Rep}\, G_A \to B$. For relations of properties of F and f see Saavedra [Sa], II, 4.3.2.

Denote by U_σ the kernel of the projection $G_\sigma \to G_\sigma^{\mathrm{red}}$; it is pro-unipotent. By the Levi decomposition, the extension $1 \to U_\sigma \to G_\sigma \to G_\sigma^{\mathrm{red}} \to 1$ splits. More precisely, the essentially surjective functor (a functor is called *essentially surjective* if each object in the target category is isomorphic to an object in the image of the functor) $\mathrm{Gr}^W : \mathcal{M}_k \to \mathcal{M}_k^{\mathrm{red}}$, defined on one-motives by

$$M = (X, A, T, E, u) \mapsto H(X) \oplus H(A) \oplus H(X_*(T))(1),$$

is an inverse to $\mathcal{M}_k^{\mathrm{red}} \hookrightarrow \mathcal{M}_k$. Correspondingly $G_\sigma^{\mathrm{red}} = \mathrm{Aut}^{\otimes}(H_\sigma^{\#} \otimes k|\mathcal{M}_k^{\mathrm{red}} \otimes k)$ is canonically a subgroup of $\mathrm{Gr}^W G_\sigma = \mathrm{Aut}^{\otimes}(H_\sigma^{\#} \mathrm{Gr}^W \otimes k|\mathcal{M}_k \otimes k)$, which is isomorphic by the Levi decomposition — but not canonically — to $G_\sigma = \mathrm{Aut}^{\otimes}(H_\sigma^{\#} \otimes k|\mathcal{M}_k \otimes k)$.

Our main object of study is the affine scheme

$$P_\sigma = \mathrm{Hom}^{\otimes}(H_{\mathrm{DR}}^{\#}, H_\sigma^{\#} \otimes k; \mathcal{M}_k \otimes k)$$

over k of morphisms of fiber functors ([DM], Theorem 3.2). It is a G_σ-torsor (right principal homogeneous space) over k, and so it defines a class h_σ of the first Galois cohomology set $H^1(k, G_\sigma) = H^1(\mathrm{Gal}(\overline{k}/k), G_\sigma(\overline{k}))$. The group G_σ is called the (σ-)*motivic Galois group* of $\mathcal{M}_k \otimes k$, and P_σ the (σ-)*motivic torsor* of $\mathcal{M}_k \otimes k$. Analogously we have the G_σ^{red}-torsor

$$P_\sigma^{\mathrm{red}} = \mathrm{Hom}^{\otimes}(H_{\mathrm{DR}}^{\#}, H_\sigma^{\#} \otimes k; \mathcal{M}_k^{\mathrm{red}} \otimes k)$$

over k, and its class h_σ^{red} in $H^1(k, G_\sigma^{\mathrm{red}})$. The G_σ^{red}-torsor P_σ^{red} is the quotient P_σ/U_σ.

Denote by $\mathcal{M}_k^0$ the Tannakian subcategory generated by Artin motives $[X \to 0]$ in $\mathcal{M}_k$. It is equivalent to the category of [DM], Proposition 6.17, generated by the zero dimensional varieties Z over k. The motivic Galois group $\mathrm{Aut}^{\otimes}(H_\sigma^{\#} \otimes k |\ \mathcal{M}_k^0 \otimes k)$ of $\mathcal{M}_k^0 \otimes k$ is the constant pro-finite group scheme $\Gamma_k = \varprojlim_K \coprod_\gamma (\operatorname{Spec} k)_\gamma$ $[(k \subset) K$ finite Galois extensions, $\gamma \in \mathrm{Gal}(K/k)]$ over k (with structure morphisms $\coprod_{\gamma \in \mathrm{Gal}(K/k)} \mathrm{id}_\gamma$). Its group of $\overline{k}$-points is $\mathrm{Gal}(\overline{k}/k)$, and the functor $H_\sigma^{\#} \otimes k$ $(: X \mapsto X \otimes k$, or $: Z \mapsto k^{Z(\overline{k})}$ in [DM], 6.17) induces an isomorphism $\mathcal{M}_k^0 \otimes k \tilde{\to} \mathrm{Rep}_k(\Gamma_k)$ ([DM], Proposition 6.17). The group Γ_k is the group of connected components of G_σ^{red} ([DM], Proposition 6.23(a,b)). [Note that the proofs of Propositions 6.22(a), 6.23 of [DM] are incorrect for the full category of pure motives as stated there, but they do apply in our context of motives of abelian varieties and one-motives; see Remark 1 at the end of this paper.]

Thus the inclusion $\mathcal{M}_k^0 \hookrightarrow \mathcal{M}_k^{\mathrm{red}}$ defines a surjection $G_\sigma^{\mathrm{red}} \stackrel{\pi}{\twoheadrightarrow} \Gamma_k$ (by [DM], Remark 2.29). Its kernel $G_\sigma^{\mathrm{red},0}$ is the connected component of the identity of G_σ^{red}, a connected pro-reductive affine k-group scheme which is the motivic Galois group $\mathrm{Aut}^{\otimes}(H_\sigma^{\#} \otimes k | \mathcal{M}_{\overline{k}}^{\mathrm{red}} \otimes k)$ of $H_\sigma^{\#} \otimes k$ on $\mathcal{M}_{\overline{k}}^{\mathrm{red}} \otimes k$. The almost surjective functor (we say that a functor is *almost surjective* if each object of the target category is isomorphic to a subquotient of an object in the image of the functor; see [DM], Proposition 2.21(b)) $\mathcal{M}_k^{\mathrm{red}} \to \mathcal{M}_{\overline{k}}^{\mathrm{red}}$, $A \mapsto \overline{A} = A \otimes_k \overline{k}$, defines the injection $G_\sigma^{\mathrm{red},0} \stackrel{\iota}{\hookrightarrow} G_\sigma^{\mathrm{red}}$. In particular, denote the quotient p: $P_\sigma^{\mathrm{red}} \to P_\sigma^{\mathrm{red}}/G_\sigma^{\mathrm{red},0}$ by P_σ^{Art}. It is the Γ_k-torsor $\mathrm{Hom}^{\otimes}(H_{\mathrm{DR}}^{\#}, H_\sigma^{\#} \otimes k; \mathcal{M}_k^0 \otimes k)$. Its class h_σ^{Art} in $H^1(k, \Gamma_k) = H^1(\mathrm{Gal}(\overline{k}/k), \Gamma_k(\overline{k}))$ is the image of $h_\sigma = \{P_\sigma^{\mathrm{red}}\}$ under the map $H^1(k, G_\sigma^{\mathrm{red}}) \to H^1(k, \Gamma_k)$.

Since G_σ is the semidirect product of the pro-reductive G_σ^{red} and the pro-unipotent U_σ, we have that Γ_k is the group of connected components of G_σ. The inclusion $\mathcal{M}_k^0 \to \mathcal{M}_k$ defines a surjection $G_\sigma \stackrel{\pi}{\to} \Gamma_k$ ([DM], Proposition 2.21(a)), whose kernel G_σ^0 is the connected component of the identity of G_σ. This connected affine k-group scheme is the motivic Galois group $\mathrm{Aut}^{\otimes}(H_\sigma^{\#} \otimes k | \mathcal{M}_{\overline{k}} \otimes k)$ of $H_\sigma^{\#} \otimes k$ on $\mathcal{M}_{\overline{k}} \otimes k$. The almost surjective functor $\mathcal{M}_k \to \mathcal{M}_{\overline{k}}$, defined on one-motives by $M \mapsto \overline{M} = M \otimes_k \overline{k} = [X \stackrel{u}{\to} E \otimes_k \overline{k}]$, induces the injection $G_\sigma^0 \stackrel{\iota}{\to} G_\sigma$. The quotient p: $P_\sigma \to P_\sigma/G_\sigma^0$ is P_σ^{Art}. Its class in $H^1(k, \Gamma_k)$ is the image of $h_\sigma = \{P_\sigma\}$ under the map $H^1(k, G_\sigma) \to H^1(k, \Gamma_k)$. The functor $H_\sigma \otimes k$ maps the Tannakian category $\mathcal{M}_{\overline{k}} \otimes k$ to the Tannakian category of k-mixed Hodge structures. This would help us understand what we need to know about our motivic objects, but this map is not fully faithful when $k \neq \mathbb{Q}$.

The statement of our theorem uses the set $\operatorname{Sper} k$ of orderings ξ of the field

k. It is a compact totally disconnected topological space, where a basis of the topology is given by the sets $\{\xi;\, a > 0 \text{ in } \xi\}$ for all a in k (see, e.g., Scharlau [Sc], Ch. 3, §5). The space $\operatorname{Sper} k$ is naturally homeomorphic to the quotient of the space $\operatorname{Inv}(\operatorname{Gal}(\overline{k}/k))$ of involutions (elements of order precisely two) in $\operatorname{Gal}(\overline{k}/k)$ (endowed with the usual profinite topology) by conjugation under $\operatorname{Gal}(\overline{k}/k)$. Denote by k_ξ a real closure of k (in $\overline{k} \subset \mathbb{C}$) whose ordering induces ξ on k. Then $\operatorname{Gal}(\overline{k}/k_\xi)$ is generated by c_ξ in $\operatorname{Inv}(\operatorname{Gal}(\overline{k}/k))$. If c is an involution in $\operatorname{Gal}(\overline{k}/k)$, for any field k, then $\operatorname{char} k = 0$, the fixed field of c in $\overline{k}$ is a real closure k_ξ of k whose ordering induces ξ on k, $\overline{k} = k_\xi(\sqrt{-1})$, and the restriction of c to the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ is non-trivial (it takes $\sqrt{-1}$ to $-\sqrt{-1}$), hence it is in the unique conjugacy class of involutions in $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The ordered field k (or k_ξ) embeds in a real closed field R_ξ of codimension 2 in $\mathbb{C}$ — thus $\mathbb{C} = R_\xi(\sqrt{-1})$ — whose ordering induces ξ on k. An ordering ξ of k is called *archimedean* if the real closure k_ξ embeds in $\mathbb{R}$. When k is finitely generated, the set $\operatorname{Arch} k$ of archimedean orderings in k is dense in $\operatorname{Sper} k$; this is shown below.

The affine k_ξ-scheme $P_{\sigma,\xi} = P_\sigma \otimes_k k_\xi$ is a G_σ-torsor over k_ξ. Its class $h_{\sigma,\xi}$ in $H^1(k_\xi, G_\sigma) = H^1(\operatorname{Gal}(\overline{k}/k_\xi), G_\sigma(\overline{k}))$ is the image of h_σ under the natural localization map $H^1(k, G_\sigma) \to H^1(k_\xi, G_\sigma)$. Alternatively it can be described using the fact that the natural map $H^1(k_\xi, G_\sigma) \to H^1(R_\xi, G_\sigma)$ is an isomorphism (this is implied by the Artin–Lang theorem (see [BCR], Théorème 4.1.2)), as follows. The continuous map $\sigma M(\mathbb{C}) \to \sigma M(\mathbb{C})$ ($M \in \mathcal{M}_k$) defined by $c_\xi \neq 1$ in $\operatorname{Gal}(\mathbb{C}/R_\xi)$ induces an involutive endomorphism of $\sigma M(\mathbb{C})_B$. The image in $G_\sigma(\mathbb{C})$ defines a (Galois) cohomology class in $H^1(R_\xi, G_\sigma)$, which is $h_{\sigma,\xi}$.

Let k be a field with virtual cohomological dimension ≤ 1 (thus $\operatorname{vcd}(k) = \operatorname{cd}(k(\sqrt{-1}))$ is at most one). We have $\operatorname{vcd}(k) = \operatorname{cd}(k)$ precisely when k has no orderings, thus $\operatorname{Sper} k$ is empty. Examples of k with $\operatorname{vcd} k = 1 < \operatorname{cd} k$ are $k = \mathbb{R}(x)$ or $R(x)$, where R is a real closed field (Serre [S1], II, 3.3(b)), $\mathbb{R}((x))$ and $R((x))$ ([S1], II, 3.3, Ex. 3), and $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$ ([S1], II, 3.3, Proposition 9). We assume that k embeds in $\mathbb{C}$ (to use [DM]; to embed in $\mathbb{C}$ a field k of cardinality bounded by that of $\mathbb{C}$, choose transcendence bases in both). Fix $\sigma\colon k \hookrightarrow \mathbb{C}$.

THEOREM: *Let $p'\colon P' \to P_\sigma^{\mathrm{Art}}$ be a G_σ-torsor over k such that $P'_\xi = P' \otimes_k k_\xi$ is isomorphic to $P_{\sigma,\xi} = P_\sigma \otimes_k k_\xi$ for ξ in a dense subset of* $\operatorname{Sper} k$. *Then there exists an isomorphism of G_σ-torsors $\lambda\colon P_\sigma \to P'$ such that $p' \circ \lambda = p$.*

The same result holds with G_σ and P_σ replaced by G_σ^{red} and P_σ^{red}.

Our work is influenced by Blasius–Borovoi [BB] who considered the number field $\mathbb{Q}$ (whose vcd is 2) and the semisimple Tannakian subcategory $\mathcal{M}_{\mathbb{Q}}^{\mathrm{red},H}$ generated by Artin motives and motives of abelian varieties A over $\mathbb{Q}$ for which the group $((G_\sigma^A)^0)_{\mathbb{R}}^{\mathrm{ad}}$ has no factor of type D_n^H (in the notations of Deligne [D1], (1.3.9)), and by Wintenberger [W] who had considered the field $\mathbb{Q}$ and the semisimple Tannakian subcategory $\mathcal{M}_{\mathbb{Q}}^{\mathrm{red,CM}}$ generated by Artin motives and motives of abelian varieties with complex multiplication over $\mathbb{Q}$.

Our theorem is an application of the local-global principle for a field k with $\mathrm{vcd}(k) \leq 1$. We can work in the generality of the entire category $\mathcal{M}_k$ and the group G_σ by virtue of the local-global principle: $H^1(k,G) \hookrightarrow \prod_\xi H^1(k_\xi,G)$, where ξ ranges over any dense subset of $\mathrm{Sper}\, k$, proven by Scheiderer [Sch] (for an alternative proof see Chernousov [Ch]) for a perfect field k with $\mathrm{vcd}\, k \leq 1$ and a connected k-linear algebraic group G. In the number field case the analogous well known local-global principle holds only for semisimple simply connected G.

When $\mathrm{cd}(k) \leq 1$, thus when k has no orderings, the class of P_σ is determined by P_σ^{Art} alone. To deal with this case, we use only Steinberg's theorem ([S1], III, §2.3) on the vanishing of $H^1(k,G)$ for a perfect field k with $\mathrm{cd}(k) \leq 1$ and a connected k-linear algebraic group G.

It will be interesting to study our motivic objects over fields k with $\mathrm{vcd} \leq 2$. In this context, note that a local-global principle for k with $\mathrm{vcd}(k) \leq 2$ and semisimple simply connected classical linear algebraic groups has recently been established by Bayer-Fluckiger and Parimala [BP].*

Proof of Theorem: It is easy to adapt the proof to the context of the pro-reductive quotient group G_σ^{red}, so we discuss only the general case of the entire group G_σ.

Let $z \in Z^1(k,G_\sigma)$ be a 1-cocycle representing $h_\sigma = \{P_\sigma\} \in H^1(k,G_\sigma)$. As in [S1], I.5.3, denote by ${}_zG_\sigma$ the form of G_σ twisted by z. It is the affine group scheme over k on which $\mathrm{Gal}(\overline{k}/k)$ acts by s: $g \mapsto (\mathrm{Int}(z_s))(s(g))$ $(g \in G_\sigma(\overline{k})$, $s \in \mathrm{Gal}(\overline{k}/k))$. The natural bijection $H^1(k,{}_zG_\sigma) \tilde{\to} H^1(k,G_\sigma)$, defined by $(x_s) \mapsto (x_s z_s)$ ([S1], I.5.3, Proposition 35), takes the trivial element of $H^1(k,{}_zG_\sigma)$ to h_σ. Denote by η the class in $H^1(k,{}_zG_\sigma)$ which maps to h', the class in $H^1(k,G_\sigma)$ of the G_σ-torsor P'. By the very definition of P_σ, as relating G_σ and G_{DR}, we have that G_{DR} is ${}_{P_\sigma}G_\sigma = P_\sigma \times_{G_\sigma} G_\sigma$ (this is ${}_FP = P \times^A F$ in the notations of the first paragraph of [S1], I, §5.3; here A of [S1] is $G_\sigma(\overline{k})$, which acts on $P_\sigma(\overline{k})$ (=P in [S1]) by right multiplication and on $G_\sigma(\overline{k})$ (=F in [S1]) by conjugation). By the third paragraph of [S1], I, §5.3, we have that G_{DR} is the twist ${}_zG_\sigma$ of G_σ by z. Since $P'_\xi \simeq P_{\sigma,\xi}$, the localization $\eta_\xi = \mathrm{loc}_\xi(\eta)$ of η in $H^1(k_\xi,G_{\mathrm{DR}})$ is 1, for a dense set of ξ in $\mathrm{Sper}\, k$. Since P_σ and P' project to the same Γ_k-torsor P_σ^{Art} in $H^1(k,\Gamma_k)$, the image of η in $H^1(k,{}_{z'}\Gamma_k)$ is 1, where z' in $Z^1(k,\Gamma_k)$ is the image

* *Added in proof*: This has recently been done in the preprint [F].

of $z \in Z^1(k, G_\sigma)$ under the projection $G_\sigma \to \Gamma_k$. Our aim is to show that $\eta = 1$ in $H^1(k, G_{\mathrm{DR}})$.

Consider the exact sequence of affine group schemes

$$1 \to G^0_{\mathrm{DR}} = {}_zG^0_\sigma \to G_{\mathrm{DR}} = {}_zG_\sigma \to \Gamma_{k,\mathrm{DR}} = {}_{z'}\Gamma_k \to 1.$$

Since the image of $\eta \in H^1(k, G_{\mathrm{DR}})$ in $H^1(k, \Gamma_{k,\mathrm{DR}})$ is trivial, there is $\eta^0 \in H^1(k, G^0_{\mathrm{DR}})$ which maps to η. The group G^0_{DR} is a connected pro-algebraic affine group scheme over k ([DM], Proposition 6.22(a)). Thus $G^0_{\mathrm{DR}} = \varprojlim_N (G^N_{\mathrm{DR}})^0$, where G^N_{DR} is the motivic Galois group $\mathrm{Aut}^\otimes(H^\#_{\mathrm{DR}} | \mathcal{M}^N_{k_N} \otimes k_N)$ of the Tannakian subcategory $\mathcal{M}^N_{k_N}$ of $\mathcal{M}_k$ generated by a finite set N of one-motives and their duals, the Artin motives and the Tate motive T and its dual $T^\vee$. The finite set N is defined over a finitely generated over $\mathbb{Q}$ subfield k_N of k.

As explained in the proof of [DM], Proposition 6.22(a), $(G^N_{\mathrm{DR}})^0$ is a linear algebraic group. Correspondingly $\eta^0 = \varprojlim_N \eta^0_N$, where $\eta^0_N \in H^1(k, (G^N_{\mathrm{DR}})^0)$. Further, $\eta = \varprojlim_N \eta_N$, where η_N is the image of η^0_N under the map $H^1(k, (G^N_{\mathrm{DR}})^0) \to H^1(k, G^N_{\mathrm{DR}})$. Since η_ξ is trivial in $H^1(k_\xi, G_{\mathrm{DR}})$, the localization $\eta_{N,\xi} = \mathrm{loc}_\xi(\eta_N)$ is trivial for all N, for the dense set of ξ in $\mathrm{Sper}\, k$ of the theorem.

Write $\mathrm{Arch}\, k$ for the set of archimedean orderings in $\mathrm{Sper}\, k$. The proposition below asserts that the homomorphism $G_{\mathrm{DR}}(k_\xi) \to \Gamma_{k,\mathrm{DR}}(k_\xi)$ is surjective for every $\xi \in \mathrm{Arch}\, k$. In particular $G^N_{\mathrm{DR}}(k_\xi) \twoheadrightarrow \Gamma_{k_N,\mathrm{DR}}(k_\xi) = \mathbb{Z}/2$ for each finite N and $\xi \in \mathrm{Arch}\, k$. We claim that this map is onto for all $\xi \in \mathrm{Sper}\, k$. The k_N-group G^N_{DR} has two connected components; denote by $C = G^{N,+}_{\mathrm{DR}}$ the component not containing the identity. The surjectivity means that $C(k_\xi)$ is non empty (for all $\xi \in \mathrm{Arch}\, k$). It follows from the Artin–Lang theorem that $C(k_{N,\xi})$ is non-empty for all $\xi \in \mathrm{Arch}\, k_N$. But the set of $\xi \in \mathrm{Sper}\, k_N$ such that $C(k_{N,\xi})$ is non-empty is open and closed in $\mathrm{Sper}\, k_N$ (see, e.g., [Sch], Corollary 2.2). Our claim follows once we show that for a finitely generated field k_N, the set $\mathrm{Arch}\, k_N$ is dense in $\mathrm{Sper}\, k_N$.

LEMMA 0: *For a finitely generated field k_N the set* $\mathrm{Arch}\, k_N$ *is dense in* $\mathrm{Sper}\, k_N$.

Proof of Lemma 0: Choose a purely transcendental extension $F = \mathbb{Q}(t_1, \dots, t_n)$ of $\mathbb{Q}$ of finite codimension in k_N. Since the restriction of orderings is an open map $\mathrm{Sper}\, k_N \to \mathrm{Sper}\, F$, and an ordering of k_N is archimedean if its restriction to F is, it suffices to show that $\mathrm{Arch}\, F$ is dense in $\mathrm{Sper}\, F$. For this, we proceed to show that the non-empty basic open set defined by $p_1, \dots, p_r \in F$ contains an archimedean ordering. The open set being non-empty means that there is an ordering of F which makes the p_j positive. In other words, there are a real closed field R and $x \in R^n$ such that $p_j(x) > 0$, all j. Then the same is true for $R = \mathbb{R}$, by the Tarski principle (see, e.g., [BCR], I.1.4). That is, there is $x \in \mathbb{R}^n$ such that $p_j(x) > 0$, all j. The inequalities remain true in a neighborhood of x, hence the components $x_1, \ldots, x_n$ of x can be chosen to be algebraically independent. The embedding $F \hookrightarrow \mathbb{R}$ defined by $t_i \mapsto x_i$ defines an archimedean ordering of F where the p_j are positive, namely an archimedean point in the given open set. This completes the proof of Lemma 0. ∎

We then have that $G^N_{\mathrm{DR}}(k_\xi) \twoheadrightarrow \Gamma_{k_N,\mathrm{DR}}(k_\xi) = \mathbb{Z}/2$ for each finite set N of one-motives, and for all $\xi \in \operatorname{Sper} k$. Consequently the kernel of the map $H^1(k_\xi, (G^N_{\mathrm{DR}})^0) \to H^1(k_\xi, G^N_{\mathrm{DR}})$ is trivial for all ξ. For the dense set of $\xi \in \operatorname{Sper} k$ given in the theorem, $\eta_{N,\xi}$ is trivial in $H^1(k_\xi, G^N_{\mathrm{DR}})$. Then for these ξ we have that $\eta^0_{N,\xi} = \operatorname{loc}_\xi \eta^0_N$ is trivial in $H^1(k_\xi, (G^N_{\mathrm{DR}})^0)$.

Using the local-global principle of [Sch], Theorem 4.1, which asserts that for a connected linear algebraic group G^N over a perfect field k with $\operatorname{vcd}(k) \leq 1$ the map $H^1(k, G^N) \to \prod_\xi H^1(k_\xi, G^N)$ is injective where the product ranges over any dense subset of orderings ξ in $\operatorname{Sper} k$, we conclude that η^0_N is 1 for all finite sets N of one-motives. Hence $\eta^0 = \varprojlim_N \eta^0_N$ is trivial, so is its image η, and P' and P_σ define the same class in $H^1(k, G_\sigma)$. This completes the proof of the theorem. ∎

The following lemma is used in the proof of the proposition below.

LEMMA: *Let K_ξ be a real closed field containing k_ξ. Then the group of K_ξ-points of $\Gamma_{k,\mathrm{DR}} = {}_{z'}\Gamma_k$ is isomorphic to* $\operatorname{Gal}(\overline{k}/k_\xi)$.

Proof: We have $\Gamma_{k,\mathrm{DR}}(K_\xi) = \Gamma_{k,\mathrm{DR}}(K)^{\operatorname{Gal}(K/K_\xi)}$, where $K = K_\xi(\sqrt{-1})$, and $\Gamma_{k,\mathrm{DR}}(K) = \Gamma_{k,\mathrm{DR}}(\overline{k})$. Moreover, the restriction to $\overline{k}$ of the non-trivial element of $\operatorname{Gal}(K/K_\xi)$ is the non-trivial element of $\operatorname{Gal}(\overline{k}/k_\xi)$. The group $\Gamma_{k,\mathrm{DR}}$ is the profinite group scheme attached to the identity cocycle $z'(\tau) = \tau$ in $Z^1(k, \Gamma_k)$ (this is called the Artin cocycle, see [W]). Thus $\tau \in \operatorname{Gal}(\overline{k}/k)$ acts on $\gamma \in \Gamma_{k,\mathrm{DR}}(\overline{k}) = \operatorname{Gal}(\overline{k}/k)$ by $\tau_{\mathrm{DR}}(\gamma) = \tau\gamma\tau^{-1}$. In particular $c_\xi \in \operatorname{Gal}(\overline{k}/k_\xi)$ acts on $\gamma \in \Gamma_{k,\mathrm{DR}}(\overline{k})$ by $c_{\xi,\mathrm{DR}}(\gamma) = c_\xi\gamma c_\xi^{-1}$. Hence $\Gamma_{k,\mathrm{DR}}(k_\xi) = \{\gamma \in \operatorname{Gal}(\overline{k}/k); c_\xi\gamma c_\xi^{-1} = \gamma\}$. It remains to determine the centralizer of $c_\xi \in \operatorname{Inv}(\operatorname{Gal}(\overline{k}/k))$ in $\operatorname{Gal}(\overline{k}/k)$. We claim it is $\{1, c_\xi\}$. The field $k_\xi = \overline{k}^{c_\xi}$ of fixed points of c_ξ in $\overline{k}$ is a real closure of k whose ordering induces ξ on k. If $\gamma \in \operatorname{Gal}(\overline{k}/k)$ commutes with c_ξ then it maps k_ξ to itself. But the only automorphism of k_ξ over k is the identity (by the Artin–Schreier theorem; see, e.g., [Sc], Ch. 3, Theorem 2.1). Hence $\gamma \in \operatorname{Gal}(\overline{k}/k_\xi) = \{1, c_\xi\}$. ∎

The following proposition is used in the proof of the Theorem above.

PROPOSITION: *The map* $G_{\rm DR}(k_\xi) \to \Gamma_{k,{\rm DR}}(k_\xi)$ *is surjective for every archimedean ordering* ξ *in* Sper k.

Proof: The lemma implies that $\Gamma_{k,{\rm DR}}(k_\xi) = \mathbb{Z}/2 = \Gamma_{k_\xi,{\rm DR}}(k_\xi)$. Write $G_{k,\sigma}$ and $G_{k,{\rm DR}}$ to specify the base field. Using the functor $\mathcal{M}_k \to \mathcal{M}_{k_\xi}$ which is induced from $M \mapsto M \otimes_k k_\xi$ (incidentally, it is almost surjective (by which we mean that each object of $\mathcal{M}_{k_\xi}$ is a subquotient of an object in the image of $\mathcal{M}_k$), by the proof of [DM], 6.23 (a)), we have a k_ξ-homomorphism $G_{k_\xi,{\rm DR}} \to G_{k,{\rm DR}}$ (in fact an injection, by [Sa], II, 4.3.2 g) ii), or [DM], Proposition 2.21 (b)) of the motivic Galois groups for the de Rham fiber functor. Hence it suffices to prove the proposition only for a real closed k. Since ξ is archimedean, k embeds in $\mathbb{R}$, and it suffices to prove the proposition for $k = \mathbb{R}$. Thus we assume from now on that k is $\mathbb{R}$, and write $G_{\rm DR}$ for $G_{\mathbb{R},{\rm DR}}$.

Recall that the functors $\mathcal{M}_{\mathbb{R}}^0 \to \mathcal{M}_{\mathbb{R}}$ and $\mathcal{M}_{\mathbb{R}} \to \mathcal{M}_{\mathbb{C}}$, and the fiber functor $H_\sigma^\#$, define the exact sequence $1 \to G_\sigma^0 \to G_\sigma \to \Gamma_{\mathbb{R}} \to 1$ of affine group schemes over $\mathbb{Q}$ (for the "pure" case, which implies at once the "mixed" case, see [DM], Proposition 6.23(a,b)). Using the functors $\mathcal{M}_{\mathbb{R}}^0 \otimes \mathbb{R} \to \mathcal{M}_{\mathbb{R}} \otimes \mathbb{R}$ and $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{R} \to \mathcal{M}_{\mathbb{C}} \otimes \mathbb{R}$, and the fiber functor $H_\sigma^\# \otimes \mathbb{R}$, the groups become groups over $\mathbb{R}$ (note that [DM], Remark 3.12, applies with any — not necessarily finite — field extension k'/k). But we do not change the notations.

For any subfield K of $\mathbb{R}$, a *K-Hodge structure* ("over $\mathbb{C}$") is a pair $(V, (V^{p,q}))$ consisting of a finite dimensional vector space V over K, and a direct sum decomposition $V \otimes_K \mathbb{C} = \oplus V^{p,q}$ with $\tau_\infty(V^{p,q}) = V^{q,p}$; $\tau_\infty \neq 1$ in ${\rm Gal}(\mathbb{C}/\mathbb{R})$. A *K-Hodge structure over* $\mathbb{R}$ is a triple $(V, (V^{p,q}), F_\infty)$ where the new ingredient is an involutive endomorphism F_∞ of V whose extension to $V \otimes_K \mathbb{C}$ satisfies $F_\infty(V^{p,q}) = V^{q,p}$. With the natural definition of tensor products and morphisms, these make neutral Tannakian categories ${\rm Hod}_K$ (K-Hodge structures) and ${\rm Hod}_K^+$ (K-Hodge structures over $\mathbb{R}$) over K (for the forgetful fiber functor $\omega_K : (V, \ldots) \to V$).

A *K-mixed Hodge structure* ("over $\mathbb{C}$") is a triple $(V, W_\bullet, F^\bullet)$, where V is a finite dimensional K-vector space with a finite increasing (weight) filtration $W_\bullet$ and a finite decreasing (Hodge) filtration $F^\bullet$ on $V \otimes_K \mathbb{C}$, such that $F^\bullet$ induces a K-Hodge structure of weight n on the graded piece ${\rm Gr}_n^W V = W_n V / W_{n-1} V$ for each n. A *K-mixed Hodge structure over* $\mathbb{R}$ is a K-mixed Hodge structure $(V, W_\bullet, F^\bullet)$ with a $W_\bullet$ preserving involutive automorphism F_∞ of V such that $F_\infty(({\rm Gr}_n^W V \otimes_K \mathbb{C})^{p,q}) = ({\rm Gr}_n^W V \otimes_K \mathbb{C})^{q,p}$. With the natural definition of $\otimes$ and morphisms, these make the Tannakian categories ${\rm MHS}_K$ and ${\rm MHS}_K^+$.

The main Theorem 2.11 of [D2] asserts that for an algebraically closed subfield $\mathfrak{K}$ of $\mathbb{C}$, the functor H_σ: $\mathcal{M}_{\mathfrak{K}}^{\text{red}} \to \text{Hod}_{\mathbb{Q}}$ is fully faithful. It is extended in [D-III], 10.1.3, to assert that the functor H_σ: $M \mapsto H_\sigma(M) = \sigma M(\mathbb{C})_B$ defines an equivalence between the category of isogeny classes of one-motives over $\mathfrak{K}$ and the category of ($\mathbb{Q}$-)mixed Hodge structures of type $\{(0,0), (0,-1), (-1,0), (-1,-1)\}$ whose graded quotient Gr_{-1} is polarizable. A morphism of one-motives is a morphism (α,β) of complexes $[X \to E] \to [X' \to E']$. It is an *isogeny* if both α and β are isogenies, i.e., have finite kernels and cokernels. The functor H_σ extends to a faithful functor from the Tannakian category $\mathcal{M}_{\mathbb{C}}$ to the Tannakian category $\text{MHS}_{\mathbb{Q}}$ (in this context we note Theorem 2.2.5 of [Br], which asserts that a Hodge cycle on a one-motive — and in particular a power thereof — is absolute), and from $\mathcal{M}_{\mathbb{R}}$ to $\text{MHS}_{\mathbb{Q}}^+$: $\tau_\infty \in \text{Gal}(\mathbb{C}/\mathbb{R})$ induces an involution of $\sigma M(\mathbb{C})$, hence an involution $F_\infty = H_\sigma(\tau_\infty)$ on $H_\sigma(M)$. The restriction of H_σ to $\mathcal{M}_{\mathbb{R}}^0$ is an equivalence with the category $\text{Rep}_{\mathbb{Q}}\,\Gamma_{\mathbb{R}}$ of representations of $\Gamma_{\mathbb{R}}$ over $\mathbb{Q}$.

The fiber functor $H_\sigma^\# \otimes \mathbb{R}$ on $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{R}$ factorizes through the forgetful functor $\omega_{\mathbb{R}}$: $\text{MHS}_{\mathbb{R}}^+ \to \text{Rep}_{\mathbb{R}}\,\Gamma_{\mathbb{R}}$. The restriction of $\omega_{\mathbb{R}}$ to $\text{MHS}_{\mathbb{R}}$ is the forgetful functor into the category $\text{Vec}_{\mathbb{R}}$ of vector spaces over $\mathbb{R}$. The restriction of $H_\sigma^\# \otimes \mathbb{R}$ to $\mathcal{M}_{\mathbb{R}}^0 \otimes \mathbb{R}$ is an equivalence of categories with $\text{Rep}_{\mathbb{R}}\,\Gamma_{\mathbb{R}}$.

But we are concerned with the fiber functor $H_{\text{DR}}^\# \otimes \mathbb{R}$ on $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{R}$ and the exact sequence $1 \to G_{\text{DR}}^0 \to G_{\text{DR}} \to \Gamma_{\mathbb{R},\text{DR}} \to 1$ of real groups associated with the almost surjective functor $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{R} \to \mathcal{M}_{\mathbb{C}} \otimes \mathbb{R}$ and the fully faithful functor $\mathcal{M}_{\mathbb{R}}^0 \otimes \mathbb{R} \to \mathcal{M}_{\mathbb{R}} \otimes \mathbb{R}$. To show the surjectivity of the map $G_{\text{DR}}(\mathbb{R}) \to \Gamma_{\mathbb{R},\text{DR}}(\mathbb{R})$ of groups of real points, it suffices to show that the reductive part $G_{\text{DR}}^{\text{red}}(\mathbb{R})$ surjects on $\Gamma_{\mathbb{R},\text{DR}}(\mathbb{R})$. For this, note that the functor $H_{\text{DR}}^\# \otimes \mathbb{R}$ on $\mathcal{M}_{\mathbb{R}}^{\text{red}} \otimes \mathbb{R}$ factorizes via $\mathcal{M}_{\mathbb{R}}^{\text{red}} \otimes \mathbb{R} \to \text{Hod}_{\mathbb{R}}^+$ and a functor $\omega_{\text{DR},\mathbb{R}}$: $\text{Hod}_{\mathbb{R}}^+ \to \text{Vec}_{\mathbb{R}}$ described below. This follows from the fact that for the realizations of a motive one has $c_{\text{DR}} = F_\infty \circ \text{bar}$, where c_{DR} and bar are respectively the deRham and the Betti complex conjugations. Defining $\mathbb{S}_{\text{DR}}^+ = \text{Aut}^\otimes(\omega_{\text{DR},\mathbb{R}} | \text{Hod}_{\mathbb{R}}^+)$ (and $\mathbb{S}_{\text{DR}} = \text{Aut}^\otimes(\omega_{\text{DR},\mathbb{R}} | \text{Hod}_{\mathbb{R}})$), we get the vertical arrow in the commutative square

$$\begin{array}{ccc} \mathbb{S}_{\text{DR}}^+ & \longrightarrow & \Gamma_{\mathbb{R},\text{DR}} \\ \big\downarrow & & \big\| \\ G_{\text{DR}}^{\text{red}} & \longrightarrow & \Gamma_{\mathbb{R},\text{DR}}. \end{array}$$

The horizontal arrows result from the fully faithful functors $\mathcal{M}_{\mathbb{R}}^0 \otimes \mathbb{R} \to \mathcal{M}_{\mathbb{R}}^{\text{red}} \otimes \mathbb{R}$ and $\mathcal{M}_{\mathbb{R}}^0 \otimes \mathbb{R} \to \text{Hod}_{\mathbb{R}}^+$. Consequently it suffices to show that $\mathbb{S}_{\text{DR}}^+(\mathbb{R}) \twoheadrightarrow \Gamma_{\mathbb{R},\text{DR}}(\mathbb{R})$.

Analogously we have the functor $\omega_{\mathbb{R}}$ on $\mathrm{Hod}_{\mathbb{R}}^{+}$, the real groups

$$\mathbb{S}^{+} = \mathrm{Aut}^{\otimes}(\omega_{\mathbb{R}} | \mathrm{Hod}_{\mathbb{R}}^{+}) \quad \text{and} \quad \mathbb{S} = \mathrm{Aut}^{\otimes}(\omega_{\mathbb{R}} | \mathrm{Hod}_{\mathbb{R}}),$$

and the exact sequence $1 \to \mathbb{S} \to \mathbb{S}^{+} \to \Gamma_{\mathbb{R}} \to 1$. The motivic Galois group $\mathbb{S}$ of $\mathrm{Hod}_{\mathbb{R}}$ and the functor $\omega_{\mathbb{R}}$ is well known ([DM], Example 2.31). The group $\mathbb{S}$ is the connected $\mathbb{R}$-group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m$ obtained from the multiplicative group $\mathbb{G}_m$ on restricting scalars from $\mathbb{C}$ to $\mathbb{R}$. Thus $\mathbb{S}(\mathbb{C}) = \mathbb{C}^{\times} \times \mathbb{C}^{\times}$, and the non-trivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{S}(\mathbb{C})$ by $(a,b) \mapsto (\bar{b}, \bar{a})$, so $\mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$. Indeed, a representation $\rho\colon \mathbb{S} \to \mathrm{Aut}(V)$ defines $V^{p,q}$ to be the $v \in V \otimes_{\mathbb{R}} \mathbb{C}$ with $\rho(z)(v) = z^{-p}\bar{z}^{-q}v$ for all $z \in \mathbb{C}^{\times}$. The motivic Galois group of the subcategory $\mathrm{Hod}_{\mathbb{R}}^{0}$ of the V in $\mathrm{Hod}_{\mathbb{R}}^{+}$ with $V^{p,q} = \{0\}$ unless $p = q = 0$ is the constant group scheme $\Gamma_{\mathbb{R}}$ over $\mathbb{R}$ associated to the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The motivic Galois group of $\mathrm{Hod}_{\mathbb{R}}^{+}$ (and $\omega_{\mathbb{R}}$) is an extension $\mathbb{S}^{+}$ of $\Gamma_{\mathbb{R}}$ by $\mathbb{S}$. Indeed, a triple $(V, (V^{p,q}), F_{\infty})$ is associated with the extension of ρ from $\mathbb{S}$ to $\mathbb{S}^{+}$ by $\rho(1 \times \mathrm{bar}) = F_{\infty}$ ("bar" signifies complex conjugation). The exact sequence $1 \to \mathbb{S} \to \mathbb{S}^{+} \to \Gamma_{\mathbb{R}} \to 1$ is defined by the fully faithful functor $\mathrm{Hod}_{\mathbb{R}}^{0} \to \mathrm{Hod}_{\mathbb{R}}^{+}$ and the essentially surjective "forget F_{∞}" functor $\mathrm{Hod}_{\mathbb{R}}^{+} \to \mathrm{Hod}_{\mathbb{R}}$. Note that the sequence is split, and $\mathbb{S}^{+} = \mathbb{S} \rtimes \Gamma_{\mathbb{R}}$. A splitting is given by the essentially surjective functor $\mathrm{Hod}_{\mathbb{R}}^{+} \to \mathrm{Hod}_{\mathbb{R}}^{0}$, "forget the Hodge structure", and $\Gamma_{\mathbb{R}}$ acts on $\mathbb{S}$ via the Galois action.

Since $H^1(\mathbb{R}, \mathbb{S}) = 1$, the sequence $1 \to \mathbb{S}(\mathbb{R}) \to \mathbb{S}^{+}(\mathbb{R}) \to \Gamma_{\mathbb{R}}(\mathbb{R}) \to 1$ is exact. Since the group $\mathbb{S}_{\mathrm{DR}}$ is $\mathbb{G}_m^2$ (see the following paragraph), by Hilbert Theorem 90 we have $H^1(\mathbb{R}, \mathbb{S}_{\mathrm{DR}}) = 1$. Hence $\mathbb{S}_{\mathrm{DR}}^{+}(\mathbb{R}) \to \Gamma_{\mathbb{R},\mathrm{DR}}(\mathbb{R})$, which is just $H^0(\mathbb{R}, S_{\mathrm{DR}}^{+}) \to H^0(\mathbb{R}, \Gamma_{\mathbb{R},\mathrm{DR}})$, is onto. This completes the proof of the proposition.

Note that the structure of the entire group $\mathrm{Aut}^{\otimes}(\omega_{\mathrm{DR},\mathbb{R}} | \mathrm{MHS}_{\mathbb{R}})$ is computed in [D3], Construction 1.6 and Proposition 2.1, since $\omega_{\mathrm{DR},\mathbb{R}}$ is the functor Gr^W of [D3]. But by the Levi decomposition it suffices for us to work only with its reductive part. Thus we note that $\mathbb{S}_{\mathrm{DR}}^{+}$ is known to be $(\mathbb{G}_m \times \mathbb{G}_m) \rtimes \mathbb{Z}/2$. Indeed, the category $\mathrm{Hod}_{\mathbb{R}}^{+}$ is equivalent to the category $\mathrm{Hod}_{\mathbb{R}}^{*}$ of triples $(W, (W^{p,q}), F)$, where W is a finite dimensional real vector space with decomposition $W = \oplus W^{p,q}$ into real subspaces, and F is an involutive endomorphism of W over $\mathbb{R}$ with $F(W^{p,q}) = W^{q,p}$. In fact, $\mathrm{Hod}_{\mathbb{R}}^{+} \to \mathrm{Hod}_{\mathbb{R}}^{*}$ is given by $W^{p,q} =$ fixed points of $F_{\infty} \circ \mathrm{bar}$ in $V^{p,q}$, $F = F_{\infty}|W$, $W^{p,q} = W \cap V^{p,q}$, and $\mathrm{Hod}_{\mathbb{R}}^{*} \to \mathrm{Hod}_{\mathbb{R}}^{+}$ by: $V =$ fixed points of $F \circ \mathrm{bar}$ in $W \otimes \mathbb{C}$, $V^{p,q} = V \cap (W^{p,q} \otimes \mathbb{C})$, $F_{\infty} = F|V$. The fiber functor $H_{\mathrm{DR}}^{\#} \otimes \mathbb{R}$ on $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{R}$ factorizes through the fiber functor ω_{DR} on $\mathrm{Hod}_{\mathbb{R}}^{+}$, which is $V \mapsto W$, or $W \mapsto W$ on $\mathrm{Hod}_{\mathbb{R}}^{*}$. The group of automorphisms of ω_{DR} on $\mathrm{Hod}_{\mathbb{R}}^{*}$ is $\mathbb{S}_{\mathrm{DR}}^{+} = (\mathbb{G}_m \times \mathbb{G}_m) \rtimes \mathbb{Z}/2$, the product of the finite group scheme

$\Gamma_{\mathbb{R},\mathrm{DR}} = \mathbb{Z}/2$ by $\mathbb{S}_{\mathrm{DR}} = \mathbb{G}_m \times \mathbb{G}_m$, the groups of automorphisms of the functor ω_{DR} on the categories $\mathrm{Hod}^0_{\mathbb{R}}$ and $\mathrm{Hod}_{\mathbb{R}}$. ∎

Remark 1: Proposition 6.22(b) of [DM] is wrong ($\underline{M}_k \to \underline{M}_{k'}$ there is fully faithful but not essentially surjective), but this is of no consequence for the theory. For a corrected statement and a counter example see [S2], §6. The connectedness assertion in Proposition 6.22(a) (and consequently 6.23) of [DM] — which is a consequence of the standard conjectures — is out of reach of current technology (in Deligne's opinion) in the context of the whole category of (even only pure) motives. In particular, (6.1) of [S2] should be (6.1?), and similarly for [J], Theorem 4.7, p. 50. The proof of [DM], 6.22(a), implicitly assumes that Hodge cycles are absolute. It works in our setting (of motives of abelian varieties, and one-motives) since Hodge cycles on abelian varieties are absolute, by [D2], Theorem 2.11. Thus we use [DM], 6.22(a) and 6.23, replacing $\underline{M}_k$, $\underline{M}_{\bar{k}}$ by $\mathcal{M}_k^{\mathrm{red}}$, $\mathcal{M}_{\bar{k}}^{\mathrm{red}}$ in [DM], p. 213, l. -7 to p. 216, l. -9; in particular the group $G(\sigma)$ of [DM], p. 213, l. -6 (denoted G_σ here) should be $\mathrm{Aut}^{\otimes}(H_\sigma^{\#}|\mathcal{M}_k^{\mathrm{red}})$, and in the proof of [DM], 6.22(a), X should be in $\mathcal{M}_k^{\mathrm{red}}$ (to use (I 3.4)).

Yet the full Galois group $G(\sigma)$ of [DM], 6.22(a) ($= \mathrm{Aut}^{\otimes}(H_\sigma^{\#}|\underline{M}_{\bar{k}})$) is pro-reductive (as asserted in [DM], 6.22(a)) — meaning that its connected component G^0 is the projective limit of connected reductive groups — by [DM], Remark 2.28 ("G^0 is pro-reductive iff $\mathrm{Rep}_{\mathbb{Q}}\, G(\sigma)$ is semisimple") and [DM], Proposition 6.5 ("$\underline{M}_{\bar{k}} = \mathrm{Rep}_{\mathbb{Q}}\, G(\sigma)$ is semisimple").

In an attempt to clarify the proof of [DM], 6.22(a), note that it uses the following well-known assertion. Only the special case of pure Hodge structures is used in [DM], and this suffices for our purposes too, since an algebraic group is connected if its (Levi) reductive component is. As in [DM], Proof of Proposition 2.8, let C_H be the full (Tannakian) subcategory of the category Hod of $\mathbb{Q}$-Hodge structures generated by $\mathbb{Q}(1)$ and an object H. The objects of C_H are by definition the subquotients of sums of $T = H^{\otimes m_1} \otimes (H^{\vee})^{\otimes m_2} \otimes \mathbb{Q}(1)^{\otimes m_3}$, and $a \in \mathbb{G}_m$ acts on $\mathbb{Q}(1)^{\otimes m}$ by multiplication by a^{-m}. Let ω be the fiber (forgetful) functor to the category of vector spaces over $\mathbb{Q}$. Suppose that H is a polarizable Hodge structure. Then C_H is semisimple. Write G' for the subgroup $GL(H) \times \mathbb{G}_m$ over $\mathbb{Q}$ which fixes all $(0,0)$-vectors t in every object T of C_H.

ASSERTION: *The group $G = \mathrm{Aut}^{\otimes}(\omega|C_H)$ is isomorphic to the group G'.*

Proof: A morphism $g = (g_X\colon \Phi(X) \to \Phi'(X))$ of functors Φ, Φ' on a category satisfies $\Phi'(f)g_X = g_Y\Phi(f)$ for every morphism $f\colon X \to Y$. In C_H, an endomorphism of the fiber functor ω is an element g of $GL(H) \times \mathbb{G}_m$ which — extended to $H_{\mathbb{C}} = H \otimes \mathbb{C}$ — commutes with $\omega(f)$, thus $g\omega(f) = \omega(f)g$, for every morphism $f : V \to U$ in Hod, namely with all linear maps $f\colon V \to U$ with $f(V^{p,q}) \subset U^{p,q}$. Thus for each V, g commutes with $\mathrm{Hom}_{\mathrm{Hod}}(\mathbb{Q}(0), V) = V^{0,0}$, namely it fixes $V^{0,0}$, so $g \in G'$.

Conversely, if $g \in G'$ then for any $V, U \in C_H$, g fixes $(V^\vee \otimes U)^{0,0} = \mathrm{Hom}(V, U)^{0,0}$, thus $g\colon H \to H$ commutes with every morphism $f\colon V \to U$ in Hod, so $g \in G$. ∎

Now the problem in the proof of 6.22(a) in [DM] is that for X in the Tannakian category $\underline{M}_{\mathfrak{K}}$ of motives of absolute Hodge cycles, the full subcategory C_X of $\underline{M}_{\mathfrak{K}}$ embeds via H_σ in the Tannakian category Hod of $\mathbb{Q}$-Hodge structures, but it is not a full subcategory unless each σ-Hodge cycle on X is absolute. If C_X is a full subcategory of Hod (via H_σ, namely each σ-Hodge cycle is absolute), then $G_X = \mathrm{Aut}^{\otimes}(H_\sigma | C_X)$ of [DM], 6.22(a), becomes the group G of the Assertion above, and it can be compared with G', the connected group which features in the second half of [DM], proof of 6.22(a) (and (I 3.4) there). In general, the group G_X consists of those automorphisms of the vector space $H_\sigma(X)$ which commute with each automorphism of the absolute Hodge structure $H(X)$. Not every automorphism f_σ of the Hodge structure $H_\sigma(X)$ extends to an automorphism $(f_{\mathrm{DR}}, f_\ell, f_\tau)$ of absolute Hodge structures, so the group G_X — being the commutator of absolute Hodge morphisms — may be larger than the commutator G of the larger family of σ-Hodge morphisms. The two groups are equal (and the *a-priori* possibly bigger G_X is connected) for abelian varieties X, for which Hodge cycles are absolute.

Remark 2: An extension E of an abelian variety by a torus T is commutative: (a) T is central: the action by inner automorphism of $A = E/T$ on T is trivial, because it amounts to an action on the character group, which is discrete; (b) the commutator $E \times E \to E$ has image in $T = \ker[E \to A]$, and it factors via $A \times A = E/T \times E/T \to T$ by (a); it is trivial since the image is proper and reduced in the affine T.

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References

[BP] E. Bayer-Fluckiger and R. Parimala, *Classical groups and the Hasse principle*, Annals of Mathematics **147** (1998), 651–693.

[BB] D. Blasius and M. Borovoi, *On period torsors*, in *Automorphic Forms, Automorphic Representations, and Arithmetic*, Proceedings of Symposia in Pure Mathematics **66**, Part 1, AMS, Providence, RI, 1999.

[BCR] J. Bochnak, M. Coste and M.-F. Roy, *Géométrie algébrique réelle*, Ergebnisse der Mathematik 12, Springer-Verlag, Berlin, 1987.

[Br] J.-L. Brylinski, *"1-Motifs" et formes automorphes*, in *Journées Automorphes*, Publications Mathématiques de l'Université Paris VII **15** (1983), 43–106.

[Ch] V. Chernousov, *An alternative proof of Scheiderer's theorem on the Hasse principle for principal homogeneous spaces*, Documenta Mathematica **3** (1998), 135–148.

[D-II] P. Deligne, *Théorie de Hodge, II*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **40** (1971), 5–58.

[D-III] P. Deligne, *Théorie de Hodge, III*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **44** (1975), 5–77.

[D1] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, in *Automorphic Forms, Representations and L-functions*, Proceedings of Symposia in Pure Mathematics **33**, Part II (1979), 247–290.

[D2] P. Deligne (Notes by J. Milne), *Hodge cycles on abelian varieties*, in *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics **900**, Springer-Verlag, Berlin, 1982, pp. 9–100.

[D3] P. Deligne, *Structures de Hodge mixtes réelles*, in *Motives*, Proceedings of Symposia in Pure Mathematics **55** (1994), 509–514.

[DM] P. Deligne and J. Milne, *Tannakian categories*, in *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics **900**, Springer-Verlag, Berlin, 1982, pp. 101–228.

[F] Y. Flicker, *Motivic torsors over two dimensional fields*, preprint.

[J] U. Jannsen, *Mixed motives and algebraic K-theory*, Lecture Notes in Mathematics **1400**, Springer-Verlag, Berlin, 1990.

[M] J. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, in *Automorphic Forms, Shimura Varieties, and L-functions*, Perspectives in Mathematics, Vol. I, Academic Press, Boston, MA, 1990, pp. 283–414.

[Sa] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics **265**, Springer-Verlag, Berlin, 1972.

[Sc] W. Scharlau, *Quadratic and Hermitian Forms*, Grundlehren der Mathematischen Wissenschaften 270, Springer-Verlag, Berlin, 1985.

[Sch] C. Scheiderer, *Hasse principles and approximation theorems for homogeneous spaces over fields of virtual cohomological dimension one*, Inventiones Mathematicae **125** (1996), 307–365.

[S1] J.-P. Serre, *Cohomologie Galoisienne*, Cinquième édition, Lecture Notes in Mathematics **5**, Springer-Verlag, Berlin, 1994.

[S2] J.-P. Serre, *Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques*, in *Motives*, Proceedings of Symposia in Pure Mathematics **55**, Part I (1994), 377–400.

[W] J.-P. Wintenberger, *Torseurs pour les motifs et pour les représentations p-adiques potentiellement de type CM*, Mathematische Annalen **288** (1990), 1–8.